Midterm Examination

Answer all nine questions. You should justify your answer and show all details.

1. (10 points) Evaluate the iterated integral

$$\int_0^{2\sqrt{\pi}} \int_{y/2}^{\sqrt{\pi}} \sin x^2 \, dx dy \, .$$

2. (10 points) Let D be the region bounded by the curves of $y = 12 - x^2$ and $y = x$. Evaluate

$$\iint_D x \, dA(x, y) \, .$$

3. (10 points) Consider the polar curve given by $r^2 = 5\cos 4\theta$ in the plane. How many leaves it have? Find the area of one of its leaves.

4. (15 points) Let D be the region in the first quadrant bounded by the curves $x^2 + y^2 = 4, y = \sqrt{3}$, and the y-axis. For a function f defined on D, express the double integral $\iint_D f$ as iterated integrals in (a) $dxdy$, (b) $dydx$, and (c) $drd\theta$.

5. (10 points) Let Ω be the solid bounded between the surfaces of $z = \sqrt{3x^2 + 5y^2}$ and $z = \sqrt{20 - 2x^2}$. Given its density $\delta(x, y, z) = z$, find its mass.

6. (15 points) Let T be the tetrahedron whose vertices are $(0,0,0), (1,0,0), (1,2,0)$ and $(0,0,5)$. Express

$$\iiint_T f(x, y, z) \, dV$$

as an iterated integral in (a) $dzdydx$ and (b) in $d\rho d\varphi d\theta$ the spherical coordinates.

7. (10 points) Use spherical coordinates to find the volume of the solid which is bounded below by the xy-plane, above by the cone $\varphi = \pi/6$, and on the sides by the sphere $\rho = 3$.

8. (10 points)

   (a) Let f and g be continuous on the region Ω in $\mathbb{R}^3$. Prove the inequality

$$2 \iiint_\Omega |fg| \, dV \le \alpha^2 \iiint_D f^2 \, dV + \frac{1}{\alpha^2} \iiint_D g^2 \, dV \, ,$$

   where α is a positive number. Hint: Use $(a \pm b)^2 \ge 0$.

(b) Prove

$$\iiint_\Omega |fg|\, dV \le \sqrt{\iiint_\Omega f^2\, dV}\sqrt{\iiint_\Omega g^2\, dV}\ .$$

Hint: Make a good choice of α in the first inequality (a).

9. (10 points) A region D in the plane is symmetric with respect to the origin if $(-x,-y) \in D$ if and only if $(x,y) \in D$. Let g be a continuous function in such a region satisfying $g(-x,-y) = g(x,y)$. Show that

$$\iint_D g(x,y)\, dA(x,y) = 2\iint_{D_+} g(x,y)\, dA(x,y)\ ,$$

where $D_+ = \{(x,y) : (x,y) \in D,\ y \ge 0\}$. You are not allowed to use the change of variables formula in two dimension.

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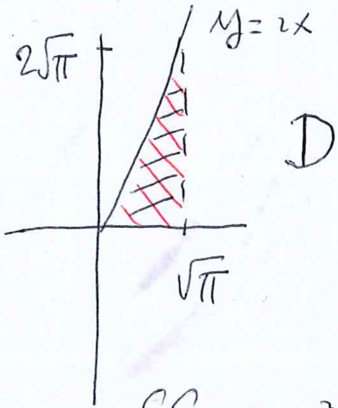
Adv. Cal. II

Solution to Midterm Exam

① $\displaystyle\int_0^{2\sqrt{\pi}}\int_{y/2}^{\sqrt{\pi}} \sin x^2 \, dx \, dy$.



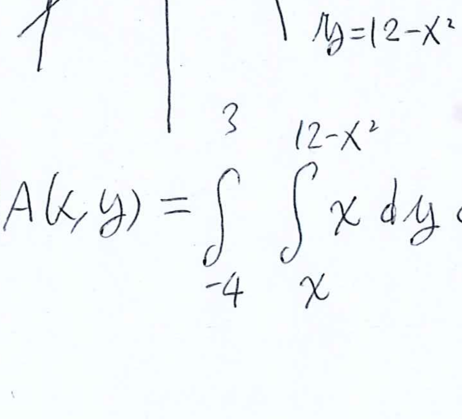
$\therefore$ the integral $\displaystyle\iint_D \sin x^2 \, dA(x,y) = \int_0^{\sqrt{\pi}}\int_0^{2x} \sin x^2 \, dy \, dx$

$\displaystyle = \int_0^{\sqrt{\pi}} 2x \sin x^2 \, dx$

$\displaystyle = -\cos x^2 \Big|_0^{\sqrt{\pi}}$

$= 2$ #

② D is



$12 - x^2 = y = x$

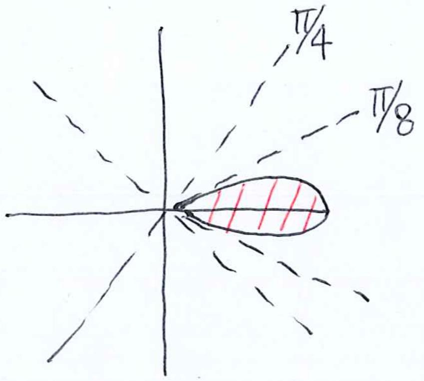
$x^2 + x - 12 = 0$

$(x+4)(x-3) = 0$

$x = -4, 3$.

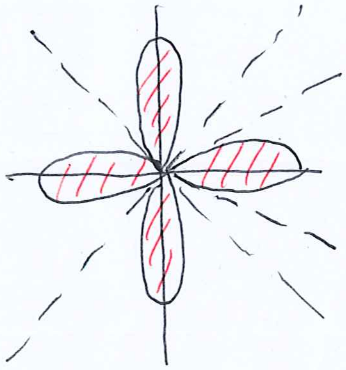
$\displaystyle\iint_D x \, dA(x,y) = \int_{-4}^{3}\int_{x}^{12-x^2} x \, dy \, dx \cdots = -\frac{343}{12}$. #

③ $\cos 4\theta$ is of ~~the~~ period $2\pi/4 = \pi/2$.
Just have to sketch its graph in $[-\pi/4, \frac{\pi}{4}]$



$\cos 4\theta \geq 0$ only on $[-\frac{\pi}{8}, \frac{\pi}{8}]$

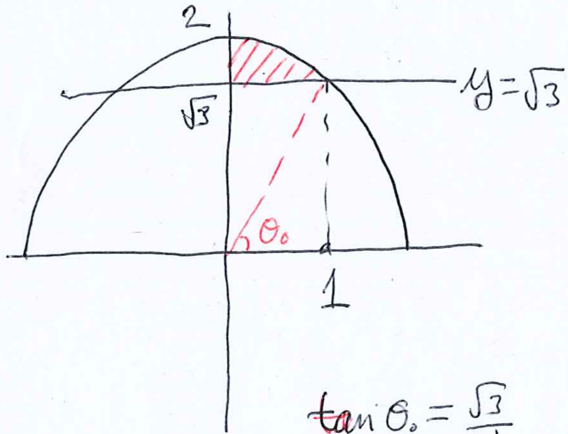
Rotate $\pi/2$, π, $3\pi/2$ to get all 4 leaves.



area of one leaf

$$= \int_{-\pi/8}^{\pi/8} \int_0^{\sqrt{5\cos 4\theta}} r\,dr\,d\theta = 5/4 \quad \#$$

④



$\tan\theta_0 = \frac{\sqrt{3}}{1}$

$\theta_0 = \pi/3$

ⓐ $\displaystyle\int_{\sqrt{3}}^{2} \int_0^{\sqrt{4-y^2}} f(x,y)\,dx\,dy$

ⓑ $\displaystyle\int_0^1 \int_{\sqrt{3}}^{\sqrt{4-x^2}} f(x,y)\,dy\,dx$

ⓒ $\displaystyle\int_{\pi/3}^{\pi/2} \int_{\sqrt{3}/\sin\theta}^{2} f(r\cos\theta, r\sin\theta)\,r\,dr\,d\theta$

⑤ Two surfaces intersect at $\sqrt{3x^2+5y^2} = \sqrt{20-2x^2}$, ie, $x^2+y^2=4$. So they are graphs over the disk D_2 :

$$x^2+y^2 \leq 4.$$

mass of Ω

$$= \iiint_{\Omega} z\,dV = \iint_{D_2} \int_{\sqrt{3x^2+5y^2}}^{\sqrt{20-2x^2}} z \, dA(x,y)$$

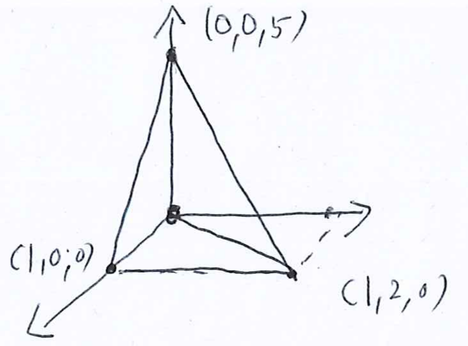
$$= \frac{1}{2} \iint_{D_2} (20 - 2x^2 - 3x^2 - 5y^2)\, dA(x,y)$$

$$= \frac{1}{2} \int_0^{2\pi} \int_0^{2} (20 - 5r^2)\, r\,dr\,d\theta$$

$$\vdots$$

$$= 20\pi \quad \#$$

(6)

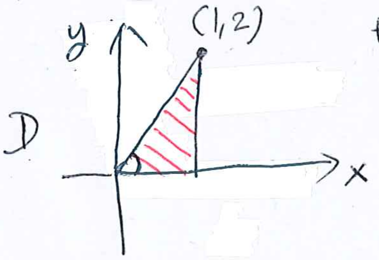


The equation of the front face is

$$5x + z = 5$$

("standard method")

The tetrahedron is the graph of $z = 5 - 5x$ over the triangle at $(0,0)$, $(1,0)$, $(1,2)$ in the xy-plane.

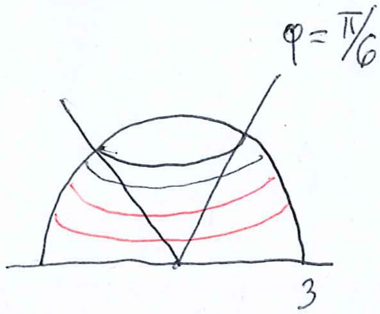


$\therefore$

(a) $$\iiint_T f\,dV = \iint_D \int_0^{5-5x} f(x,y,z)\,dz\,dA(x,y) = \int_0^1 \int_0^{2x} \int_0^{5-5x} f(x,y,z)\,dz\,dy\,dx.$$

(b) $$\iiint_T f\,dV = \int_0^{\tan^{-1}2} \int_0^{\pi/2} \int_0^{5/(5\sin\varphi\cos\theta + \cos\varphi)} f(\rho\sin\varphi\cos\theta, \rho\sin\varphi\sin\theta, \rho\cos\varphi)\,\rho^2\sin\varphi\,d\rho\,d\varphi\,d\theta.$$

⑦



$$\text{Vol} = \int_0^{2\pi}\int_{\pi/6}^{\pi/2}\int_0^{3} \rho^2 \sin\varphi \, d\rho \, d\varphi \, d\theta$$

$$= 9\sqrt{3}\,\pi \,. \#$$

⑧ (a) For $\alpha > 0$,

$$\left(\alpha\,|f(\vec{x})| - \frac{1}{\alpha}|g(\vec{x})|\right)^2 \geqslant 0, \text{ ie}$$

$$\alpha^2 f^2(\vec{x}) - 2|fg|(\vec{x}) + \frac{1}{\alpha^2} g^2(\vec{x}) \geqslant 0, \quad \forall \vec{x} \in \Omega,$$

Integrating over Ω, by linearity and positivity of the Riemann integral, key

$$\alpha^2 \iiint_\Omega f^2 - 2\iiint_\Omega |fg| + \frac{1}{\alpha^2}\iiint_\Omega g^2 \geqslant 0, \text{ ie}$$

$$\alpha^2 \iiint_\Omega f^2 + \frac{1}{\alpha^2}\iiint_\Omega g^2 \geqslant 2\iiint_\Omega |fg| \,. \#$$

(b) Choose $\alpha^2 = \sqrt{\dfrac{\iiint g^2}{\iiint f^2}}$ in (a), get

$$\iiint_\Omega |fg| \leq \sqrt{\iiint_\Omega f^2}\sqrt{\iiint_\Omega g^2}.$$

⑨ See Solution to Ex 4 for a similar problem.